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The Moduli Space of the $N = 2$ Supersymmetric G_2 Yang-Mills Theory

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Abstract

We present the hyper-elliptic curve describing the moduli space of the $N=2$ supersymmetric Yang-Mills theory with the G_2 gauge group. The exact monodromies and the dyon spectrum of the theory are determined. It is verified that the recently proposed solitonic equation is also satisfied by our solution.

In an important development in four dimensional quantum field theory, recently a number of exact results have been obtained [1]. In particular, for $N = 2$ supersymmetric gauge field theory, the exact low energy prepotential has been determined with the help of an auxiliary hyper-elliptic curve, which among other things allows the determination of the massless dyon spectrum of these theories.

This program, initially originated by Seiberg and Witten for the gauge group $SU(2)$ [2], has been carried out for $SU(N)$ [3], $SO(2n+1)$ [4] and $SO(2n)$ [5] so far. In this letter we will describe the solution for the group G_2 .

The low energy effective action for the $N = 2$ gauge theory, written in terms of the $N = 1$ fields is:

$$\frac{1}{4\pi} \text{Im} \left(\int d^4\theta \frac{\partial \mathcal{F}(A)}{\partial A^i} \bar{A}^i + \int d^2\theta \frac{1}{2} \frac{\partial^2 \mathcal{F}(A)}{\partial A^i \partial A^j} W_\alpha^i W_\alpha^j \right), \quad (1)$$

where A^i are the $N=1$ chiral field multiplets and W_α^i are the vector multiplets all in the adjoint representation. The prepotential \mathcal{F} , which at the classical level is

$$\mathcal{F} = \frac{1}{2} \tau_{cl} A^2, \quad \tau_{cl} = \frac{\theta}{2\pi} + i \frac{4\pi}{g^2}, \quad (2)$$

gets the one-loop correction

$$\mathcal{F} = \frac{i}{2\pi} A^2 \ln\left(\frac{A^2}{\Lambda^2}\right). \quad (3)$$

Λ is the dynamical scale of the theory.

The v.e.v. of the scalar component ϕ of A , determines the moduli space of the Coulomb phase of the theory, which is in turn parametrized by the invariants of the gauge group.

$$\langle \phi \rangle = \sum a_i H_i \quad (4)$$

with H_i the Cartan generators.

In a generic point of the moduli space, the gauge group will be broken to $U(1)^r$, where r is the rank of the group and all other gauge fields will become massive. However, on a singular set, there will be an enlargement of the symmetry group and correspondingly an extra set of gauge fields become massless, classically.

There are other states in the theory, the so called BPS states, with masses

$$M^2 = 2|Z|^2 = 2 \left| \sum_i q_i a_i + g_i a_i^D \right|^2 \quad (5)$$

where $a_i^D = \frac{\partial \mathcal{F}}{\partial a_i}$, which become massless on certain other singularities of the quantum moduli space. Here q_i and g_i are the electric and magnetic charges of the dyons.

The important discovery in the $N=2$ gauge theories has been the realization that the prepotentials can be described with the aid of a family of complex curves, with the identification of the v.e.v., a_i and their dual a_i^D , with the periods of the curve,

$$a_i = \oint_{\alpha_i} \lambda \quad \text{and} \quad a_i^D = \oint_{\beta_i} \lambda. \quad (6)$$

where α_i and β_i are the homology cycles of the corresponding Riemann surface.

The curves and the Riemann surfaces have been found for the A_n , B_n and D_n simple Lie groups [2-5]. The importance of the exceptional groups for the phenomenology of GUTS and string theory, and also for the possibility of a unified description of all $N=2$ gauge theories, requires knowledge of the curve for these groups. In the case of $N=1$ supersymmetric Yang-Mills theories certain results for the exceptional groups have been found, in particular for G_2 group [6]. In this letter we will present the curve for $N=2$ supersymmetric G_2 gauge theory and hope to return to the other exceptional groups in a forthcoming work.

The group G_2 is of rank two and dimension 14. We choose the two simple roots as in figure 1.

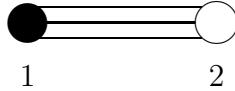


Figure 1 : G_2 Dynkin diagram

The Weyl group of G_2 is the permutation group of order 3 with inversion, generated by r_1 and r_2 ,

$$r_1 : (a_1, a_2) \rightarrow (3a_2 - a_1, a_2) \quad (7)$$

$$r_2 : (a_1, a_2) \rightarrow (a_1, a_1 - a_2) \quad (8)$$

The Casimirs of G_2 , written in term of a_1 and a_2 are:

$$\begin{aligned} u &= a_2^2 + (a_1 - a_2)^2 + (a_1 - 2a_2)^2 \\ v &= a_2^2(a_1 - a_2)^2(a_1 - 2a_2)^2 \end{aligned} \quad (9)$$

We will take the hyper-elliptic curve for G_2 to be of the form

$$y^2 = (W(x))^2 - \Lambda^{2h} x^k. \quad (10)$$

The power of Λ , twice the dual Coxeter number h , which is equal to 4 for G_2 , is determined by the $U(1)_R$ anomaly [7], and k is to be determined by the order of $W(x)$, the classical curve for G_2 .

$W(x)$ is to reflect the singularity structure of the classical theory, which is determined by the Weyl group. We will therefore construct W in such a way as to have its discriminant vanish at the Weyl chambers' walls. Proceeding in this manner, and requiring W to be Weyl invariant, we find:

$$W = (x^2 - a_2^2)(x^2 - (a_1 - a_2)^2)(x^2 - (a_1 - 2a_2)^2). \quad (11)$$

Therefore we get the curve (10) with $k = 4$. The quantum discriminant is:

$$\Delta_\Lambda = \prod_{i < j} (e_i^\pm - e_j^\pm)^2 = \Lambda^{72} \Delta^+ \Delta^- \quad (12)$$

where

$$\Delta^\pm = 64v[27(\frac{u^3}{108} - v \pm \frac{u\Lambda^4}{3})^2 - 4(\frac{u^2}{12} \mp \Lambda^4)^3]^2 \quad (13)$$

The branch cuts of the Riemann surface corresponding to the curve (10) are depicted in figure 2.

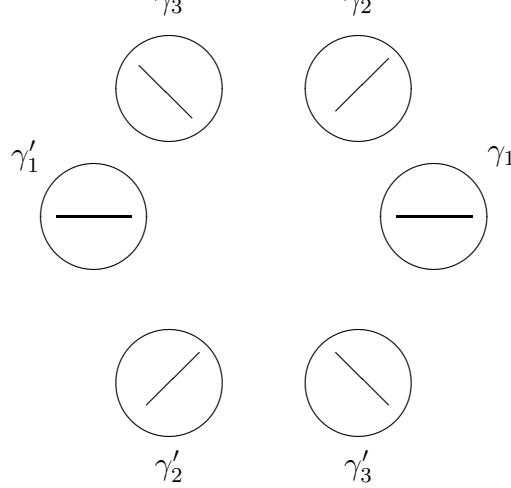


Figure 2 :Branch cuts for the curve eq. (10) and the α cycles

Where we have also indicated the corresponding cycles with γ_i and γ'_i . The two sets γ_i and γ'_i are related by the parity $x \rightarrow -x$.

To relate γ_i to α_i of the equation (6), we observe that in the limit $\Lambda \rightarrow 0$ the vanishing cycles should reproduce the classical singularities of the theory [8], leading to:

$$\begin{aligned} \gamma_1 &= \alpha_2 \\ \gamma_2 &= 2\alpha_2 - \alpha_1 \\ \gamma_3 &= \alpha_2 - \alpha_1 \end{aligned} \quad (14)$$

The intersection requirements of the cycles β_i with α_i , determine the β_i to be

$$\begin{aligned} \gamma_1^D &= \beta_2 + 2\beta_1 \\ \gamma_2^D &= -\beta_1 \end{aligned} \quad (15)$$

where γ_i^D are the conjugate cycles to γ_i

It is then straightforward to compute the monodromies for the singularity at $\Lambda \rightarrow 0$ and obtain:¹

$$B_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad B_2 = \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad B_3 = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (16)$$

¹In accordance with the notation of Ref[3]

where we have written the matrices in the (α_i, β_i) basis. By multiplying these matrices, we will obtain the quantum shift matrix T^{-1} , where

$$T = \begin{pmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & -3 & 6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (17)$$

in agreement with the shift obtained from (3) under $\Lambda^8 \rightarrow e^{2\pi i t} \Lambda^8$, $t \in [0, 1]$. The semi-classical monodromies obtained from the one-loop corrected prepotential of equation (3) are

$$\begin{aligned} M^{(r_1)} &= \begin{pmatrix} -1 & 0 & -4 & 5 \\ 3 & 1 & 7 & -9 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} & M^{(r_2)} &= \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & -1 & 3 & -4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \\ M^{(r_3)} &= \begin{pmatrix} 2 & 1 & -1 & 4 \\ -3 & -2 & 2 & -9 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 1 & -2 \end{pmatrix}, & M^{(r_4)} &= \begin{pmatrix} -2 & -1 & -1 & -2 \\ 3 & 2 & 4 & -1 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & -1 & 2 \end{pmatrix} \\ M^{(r_5)} &= \begin{pmatrix} -1 & -1 & -1 & 3 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} & M^{(r_6)} &= \begin{pmatrix} 1 & 0 & 0 & 3 \\ -3 & -1 & -3 & -1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \end{aligned} \quad (18)$$

where $r_3 = r_2 r_1 r_2^{-1}$, $r_4 = r_1 r_2 r_1^{-1}$, $r_5 = r_3 r_1 r_3^{-1}$, $r_6 = r_4 r_2 r_4^{-1}$.

To calculate the exact monodromies, we look at the vanishing cycles of the Riemann surface, as the branch points coalesce with u varying, and then use Picard-Lefshetz formula. The result is

$$\begin{aligned} M_1 &= M_{(1,0,1,-1)}, & M_2 &= M_{(1,0,-1,2)} \\ M_3 &= M_{(0,1,-1,1)}, & M_4 &= M_{(0,1,0,-1)} \\ M_5 &= M_{(1,1,0,1)}, & M_6 &= M_{(1,1,1,-2)} \\ M_7 &= M_{(3,1,1,2)}, & M_8 &= M_{(3,1,0,3)} \\ M_9 &= M_{(2,1,1,-3)}, & M_{10} &= M_{(2,1,0,-3)} \\ M_{11} &= M_{(3,2,3,-2)}, & M_{12} &= M_{(3,2,3,-3)}. \end{aligned} \quad (19)$$

In the above equation we have used the notation[9]

$$M(\mathbf{g}, \mathbf{q}) = \begin{pmatrix} \mathbb{1} - \mathbf{q} \otimes \mathbf{g} & -\mathbf{q} \otimes \mathbf{q} \\ \mathbf{g} \otimes \mathbf{g} & \mathbb{1} + \mathbf{g} \otimes \mathbf{q} \end{pmatrix}. \quad (20)$$

As a check of the curve (10), we can reproduce the semi-classical monodromies by multiplying pairs of the above exact monodromies, i.e.

$$\begin{aligned} M_2 M_1 &= M^{(r_1)}, & M_4 M_3 &= M^{(r_2)}, & M_6 M_5 &= M^{(r_3)} \\ M_8 M_7 &= M^{(r_4)}, & M_{10} M_9 &= M^{(r_5)}, & M_{12} M_{11} &= M^{(r_6)} \end{aligned} \quad (21)$$

It is interesting to note that if we consider the map S

$$\begin{pmatrix} \boldsymbol{\delta}^D \\ \boldsymbol{\delta} \end{pmatrix} = S \begin{pmatrix} \boldsymbol{\alpha}^D \\ \boldsymbol{\alpha} \end{pmatrix}. \quad (22)$$

with

$$S = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & -1 & 1 \end{pmatrix}. \quad (23)$$

then the subset $(M_1, M_2, M_5, M_6, M_9, M_{10})$ of eq. (19) monodromies, transformed as $M'_i = S^{-1} M_i S$ turn out to be those of $N=2$ gauge theory with gauge group $SU(3)$. The subset of the dyons (19) are correspondingly mapped into the dyons of the $SU(3)$ theory generated by the short roots [9, 3].

There is an interesting connection between the theories with G_2 and $SU(3)$ gauge groups which can be seen by the following change of variable in the curve of G_2 (10),

$$x^2 = x' + \frac{u}{3}, \quad \Lambda' = \Lambda^2. \quad (24)$$

giving

$$y'^2 = (x'^3 - u'x' - v')^2 - \Lambda'^4 (x' + \frac{u}{3})^2. \quad (25)$$

We recognize this curve to be that of the $N=2$ supersymmetric $SU(3)$ gauge theory with two massive hypermultiplets in the fundamental representation [10].

Our G_2 curve may shed further light on the current attempts at a better understanding of the exact results for $N = 2$ supersymmetric Yang-Mills theories. For example there have been attempts to cast these results in terms of the integrable systems [11]. In a related approach, it has been shown that, [12], the prepotential of the $N=2$ gauge theory with the gauge groups A_n, B_n and D_n satisfy the solitonic equation

$$\sum_i^r a_i \frac{\partial \mathcal{F}}{\partial a_i} - 2\mathcal{F} = 8\pi i b_1 u \quad (26)$$

where b_1 is the coefficient of the one-loop β function. To verify whether this same equation is satisfied by our curve (10), we note that equation (26) was derived from [12]

$$\sum_i^r a_i \frac{\partial \mathcal{F}}{\partial a_i} - 2\mathcal{F} = -T_1 \frac{\partial \mathcal{F}}{\partial T_1} \quad (27)$$

with the assumption that the curve is invariant under $x \rightarrow -x$. Here T_1 and $\frac{\partial \mathcal{F}}{\partial T_1}$ are the coefficient in an asymptotic expansion of the one form λ in powers of $z = \frac{1}{x}$ as $x \rightarrow \infty$

$$\lambda = \left(- \sum_{n \geq 1} n T_n z^{-n-1} + T_0 z^{-1} - \frac{1}{2\pi i} \sum_{n \geq 1} \frac{\partial \mathcal{F}}{\partial T_n} z^{n-1} \right) dz. \quad (28)$$

In the case of G_2 , λ is

$$\lambda = (-4x^6 + 2ux^4 - 2v) \frac{dx}{2\pi iy} \quad (29)$$

Reading the coefficient T_1 and $\frac{\partial \mathcal{F}}{\partial T_1}$ from the expansion of equation (29), we find

$$T_1 = \frac{4}{2\pi i}, \quad \frac{\partial \mathcal{F}}{\partial T_1} = 2u \quad (30)$$

Thus verifying the solitonic equation (26).

After completion of this work similar results appeared in the work by U. H. Danielsson and B. Sundborg.[13]

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